ANALYSIS OF THE FAILURES IN FRESHMAN MATHEMATICS.

It is my purpose to discuss in the following paper some of the causes which contribute to the failure of students to do successfully the work in freshman mathematics. This of necessity involves a discussion of the work of the public schools and in particular that done in the high schools. In this discussion, however, it must be borne in mind that the high school does not exist exclusively as a fitting school for college. It has other and perhaps more important functions to perform. It is above all a local institution, and local conditions are and should be an important factor in shaping its curriculum. It is the most democratic educational institution we possess, and offers in most cases the only opportunity the boy or girl has for a practical as well as a liberal education. Its efficiency in discharging its local obligations should be enhanced in every possible way, and any suggestion of a reform should not be given a serious consideration which cannot be shown to be in perfect harmony with this idea.

A mistake which all instructors are apt to make is to lay the shortcomings of their students at the door of the teacher or the institution charged with their preparation. Too often, also, we set up an ideal standard of proficiency which we should like our students to have upon beginning our work in order to best enable us to meet the demands laid upon *us*, and then we expect the teacher lower down the line some how to meet this condition without at the same time considering as carefully as we should whether it is possible for that teacher, under the limitations placed upon him, to do so. This results in crowding the lower strata of our educational system with too much work and often with work beyond the mental grasp of the pupil who has reached that stage of his educational development. This is perhaps nowhere more apparent than in mathematics. Any adequate discussion of high-school preparation in mathematics must

include, therefore, a discussion of college work on the one hand and grade work on the other.

One of the consequences of being a specialist and knowing intimately but one phase of a student's work is that the college professor recognizes more readily the deficiencies of students in his particular field than in any other, and often hastily concludes that they enter college with poorer preparation for the work of his department than that of others. Such a conclusion, however, cannot be accepted until we have taken into consideration all of the evidence in the case as presented by the college records. Without question the best test which we can apply to the student's preparation to do successfully the work expected of him in his freshman year at college is the record which he makes during that year. His preparation may not have been all that could be desired, but the fact that he passed and received a credit at the hands of his instructor is indisputable evidence that his preparation was, in some measure at least, satisfactory to the department. For the purpose of a comparative study, I have tabulated below, for the first semester of this year, the per cent. of failures at the University of Illinois in mathematics, rhetoric, chemistry, and first-year language, including French, German, Latin, and English. The reason for selecting these particular subjects is that they are not only representative of different lines of study, but are taken by a large number of students—and, like mathematics, are prescribed subjects for certain groups of students. The results of this comparison are as follows:

Subject.	Per cent. of failures.
1. Mathematics:	
Algebra,	15 per cent.
Trigonometry,	15 per cent.
2. Chemistry,	16 per cent.
3. Rhetoric,	10 per cent.
4. First year language:	
Latin,	No failures.
German,	19 per cent.
French,	15 per cent.
English Literature,	3 per cent.

Of these subjects, the courses in chemistry, German and French present the elements of the subject and require no previous training in the high school. Of the others, all of which

are based upon certain training obtained in the secondary schools, the largest per cent. of failures occur in mathematics. It will be urged by some that as mathematics is an exact science, the errors of the students stand out more strikingly than in the other subjects and hence are apt to be magnified in making up the final grade. It may be urged with equal fairness that for the same reason the successes stand out in greater prominence and hence are apt to be magnified in our grading. Moreover, the argument that this contributes to the large number of failures in mathematics as compared with English, for example, seems to lose its validity when a comparison is made with French and German. The question also arises as to whether the comparatively large per cent. of failures in mathematics is not peculiar to this university, due perhaps to the severity with which we grade our students. This question is answered by a consideration of the following data, obtained from the universities of Indiana, Purdue, Michigan, and Wisconsin, where, as here, a system of accredited schools is maintained, These four institutions, together with the University of Illinois, represent a total enrollment of 1,328 in college algebra and 1,065 in trigonometry. The average per cent. of failure in these five institutions was 15 in college algebra and 11 in trigonometry. From this, it will be seen that the per cent. of failures at the University of Illinois was exactly the average in algebra, while in trigonometry it is somewhat larger. This relatively large per cent. of failures in trigonometry is due, in part at least, to two causes; first, in some of the institutions mentioned trigonometry is made to follow college algebra, thus coming after the student has regained his grasp of the elementary mathematics; second, at the University of Illinois certain groups of students are permitted to take trigonometry without taking college algebra at all.

A further study of those cases where the work has proven unsatisfactory should disclose some of the causes which led to the failure. From this analysis, we obtain the following data:

FAILURES.

70 per cent. finished elementary alg. in first or second year of the high school.
22 per cent. finished elementary algebra in third year of the high school.

8 per cent. has some elementary algebra in the last year of the high school.
18 per cent. had had elementary algebra the previous year.
18 per cent. had not had elementary algebra for two years.
64 per cent. had not had elementary algebra for three or more years.
32 per cent. had spent one year or less on elementary algebra.
51 per cent. had spent *between* one and two years on elementary algebra.
17 per cent. had spent two years on elementary algebra.
None had spent more than two years on elementary algebra.

The significance of this data is more evident when we compare with it the cases where the student made an exceptionally good grade, say from 90 to 100 per cent. In a very large per cent. of these cases either the student had had some review of elementary algebra the previous year, or had spent two or more years upon the subject, or had been out of school for some time and hence was more mature and in many cases had in the meantime taught elementary algebra.

It is no doubt true that the per cent. of failures would be materially reduced if more time could be devoted in the freshman year to the study of algebra and trigonometry. This, however, is impossible in those institutions like the state universities, where engineering departments are maintained; for, in the first two years of the college course not only must algebra and trigonometry be covered, but also analytical geometry and calculus, in order that the engineering students may study during the last two years the applications of these branches to technical subjects.

The above data seem to indicate that either two or more years should be spent on the subject of algebra, or that the time spent on this subject should be so distributed that a portion of it should come as late as possible in the high-school course. Because of the obligations placed upon the high school other than that of a fitting school, it is perhaps unreasonable, if not altogether impossible, for the accredited school to meet the first of these conditions. This, however, does not apply to the second alternative. This is something which the high schools can do, and certainly should do, in order to promote in the highest degree the best interests of their pupils. The above statistics show precisely the condition of affairs which a careful

consideration would have led us to expect. Certain portions of the algebra are sufficiently abstract to be beyond the ready comprehension of pupils in the first two years of the high-school course. For example, the theory of quadratic equations, a proof of the binomial theorem, the general theory of exponents with operations involving complicated radical expressions, and simultaneous quadratic equations are illustrations of what might better be postponed until later. By attempting to teach these subjects in anything like a rigorous and satisfactory manner at this period in the high-school course, we do more serious damage than merely to waste the pupil's time. Because the mathematical principles which underlie these subjects are beyond his ready comprehension, the pupil easily comes to regard them as so many things to be taken for granted, and the teacher, finding it impossible to secure (except from the brightest of the class) anything more than mechanical work, does not insist upon more. As a consequence, the less brilliant pupil comes naturally to feel that somehow the whole thing is beyond him, and that he must have been born especially deficient in mathematical instinct. While it is doubtless true that all people could not become great mathematicians any more than they could acquire great eminence in any other line of scientific study, yet it is not too much to maintain that any pupil of average ability can acquire a reasonble knowledge of elementary mathematics, and will do so with pleasure to himself and satisfaction to his instructor, providing the subject-matter is presented to him at the right time and in legitimate doses. It is entirely possible to destroy a natural taste for mathematics by crowding the pupil forward too rapidly in his course. This is as true of the work in the grades as in the high school. In fact, the pupil's interest in any given subject depends in no small degree upon the *time* at which it is presented. This is quite as important as the *manner* in which it is presented. In my estimation, we waste much valuable time of the pupil by attempting to teach arithmetic too early in the grades. In answer to a query as to why he had the elementary number work, including the multiplication table, taught so early in the grades, a city superintendent in this state once said to

me: "They might as well be learning that as anything else, and then it will be out of the way and ready for use when they come later to take up the formal study of arithmetic." Is it possible that superintendent thought the child's mind a sort of cold-storage plant, where the facts of human knowledge could be dumped at convenient times and in any order, to be produced fresh and ready for use upon future demand? This is not sound pedagogy. It is sure to produce a chronic case of mathematical nausea, more difficult to cure than the chronic ills of a physical nature.

As to the distribution of the time devoted to mathematics in the high school, I would suggest that the first year be given to the study of the more elementary portions of the algebra. This should include enough to enable the pupil to do successfully the work in geometry and physics, say covering the work as ordinarily presented up to and including the solution of single quadratic equations having numerical coefficients. In the second year, I would recommend plane geometry, and in the third physics. In the fourth year, I would suggest a review of the fundamental operations of algebra, including more difficult exercises in factoring, theory of exponents and radicals, and a thorough drill in simultaneous quadratics and those equations of higher degree which may be solved like quadratics. This would require perhaps one-half of the last year. The remaining portion of the year might be devoted to solid geometry.

To be sure, not every subject can claim the attention of the pupil during the last year of the high-school course. Is it not reasonable, however, that subjects like mathematics, which require the highest development of the reasoning faculties, should be given a preference? On the other hand, those sciences which, as taught in the high school, are largely observational sciences, might just as well, perhaps better, come earlier. The same is true of those literary studies which have for their primary object the perfection and extension of the pupil's vocabulary either in his own or in a foreign language, and likewise of those studies, as history and civics, which aim to teach the pupil a certain definite mass of informational data.

It must be at once apparent that the pupil intending to complete a college course would be given a decided advantage by taking his preparatory training as indicated. First of all, he would enter college fresh from his high-school mathematics, and this means a much more definite mathematical stock in hand as a basis for his college work than would otherwise be the case. This is of greater importance in mathematics than in any other subject; for at most institutions, as, for example, at the University of Illinois, every student not enrolled in the professional schools is expected to take mathematics during his *freshman* year, with the single exception of certain students in the college of agriculture. This cannot be said of any other *one* subject. Furthermore, while many other subjects require only a general training and maturity as a sufficient preparation for college work, in mathematics the student must have not only this general training and development, but he must possess in addition a definite fund of technical knowledge upon which to base his subsequent work. This is the more important in mathematics because one of the general divisions of the science, namely algebra, is not completed in the secondary schools but the freshman work takes up the subject in the middle and carries it on to completion. By the arrangement suggested, not only would the continuity of the student's mathematical study be preserved, but he would enter college with a better spirit toward and a deeper interest in the work of his freshman year.

There is another reason why the above distribution of the time devoted to mathematics seems best. This concerns the pupil who does not intend to pursue a college course. His plans in this regard will be most likely matured by the time he begins his last year of the high-school course. In case he does not propose to go to a higher institution of learning, it seems to me that his time might be more profitably spent in this year by studying other subjects than the mathematics suggested. If he is to go at once into active life, he might better, perhaps, spend this time in the study of bookkeeping and commercial arithmetic, or in additional work in history, literature, or a foreign language.

It may be of interest in this connection to see how far the plan suggested is followed by the high schools of Illinois. Through the kindness of Mr. Bonzer, a fellow at the University of Illinois, I was able to obtain the necessary data from 297 high schools of the state. This list includes 95 per cent. of the high schools recognized as such by the state superintendent in his last report. Moreover, it includes all the accredited schools of the University of Illinois with the exception of five. Of these schools 70 per cent. finish the elementary algebra in the first two years, and less than 5 per cent. have any algebra in the last year. This condition of affairs seems to indicate, therefore, that some change in this regard is needed.

The completion of the mathematics too early in the high-school course is certainly not the only cause of the student's failure to do creditable work in freshman mathematics. When we study the record of our students by the schools in which the preparation was made, other causes than the one mentioned become apparent. I have endeavored as best I could at long range to investigate several of the cases where the record seemed to indicate that mistakes were being made. While the results of this study have been more or less unsatisfactory, yet enough data have been gathered to show that in some of our schools, at least, more attention should be paid to mathematics as a basis of promotion in the grades below the high school. The pupil cannot be expected to do good work in the mathematics taught in the high school who has been permitted to pass a grade where an essential part of arithmetic, as, for example, common fractions are taught, without doing the work of that grade. Mathematics is a progressive science and the connection of one topic to another is so vital that unless the pupil does each well in turn he is bound sooner or later to get beyond his depth, which means at least discouragment and most likely ultimate defeat. No matter how efficient the high-school teacher of mathematics may be, unless the work in the grades has been thoroughly done, his work is vitiated and the *pupil* reaps the consequences.

Another important element to be considered is the proper

correlation of the various branches of mathematics. We teach these branches too much as distinct subjects. As the boy passes from arithmetic to algebra, he should not feel that arithmetic is a closed book and that he is now commencing an entirely new line of study. He should be made to see the relations between the two and the applications of his algebra to his arithmetic, and to feel that he has now an additional tool at his command which will enable him to solve many of his former problems with greater ease and facility. In fact, while I should advocate the postponement of many of the topics of formal algebra as late as possible in the high-school course, I should urge that the first principles, so far as is necessary to make use of the simplest equations of the first degree, be introduced into the arithmetic. If not formally, we do in fact use the *idea* of the equation in solving many of these problems, and the introduction of the algebraic equation would not only facilitate the work of the arithmetic, but would prepare the way for and show the advantage of the formal algebra to be studied later. In the same way, after the pupil has had geometry, we should introduce into his subsequent algebra some work in the loci of equations, to the end that he might see the connection between the geometric interpretation of the equation and its analytical statement. This would not only add interest to the subject, but would give the very best preparation for his study of analytical geometry when he enters upon his college course.

Another deficiency too often noticeable in the freshman student is his inability to reason mathematically, to analyze readily a given problem and to state in mathematical terms the conditions involved. This deficiency may be due in part to the immaturity of the student, but it is doubtless also the result of a lack of training. Training of this character, whether it be in mathematics or elsewhere, is of the greatest value; for, it leads the boy to think for himself, and it is more nearly related to and is a better kind of preparation for independent research later than anything else which he does in the high school. Problems which best serve this purpose are either of the nature of supplementary propositions or the practical application of mathematics

to the solution of some problem in the physical world. This side of a mathematical training cannot be emphasized to much. It not only stimulates an interest in the subject, but makes the boy feel that he is dealing with something that is tangible and which gives him a power he did not before possess. On the other hand, the disciplinary value of the subject is increased rather than lessened by such applications. This is an additional reason why some of the high-school mathematics should follow physics.

So far in our discussion, we have not taken into consideration one important element, perhaps the most important element, in the sucessful preparation of a student for college work, namely the character and training of the teacher himself. Aside from that general training and ability which he should have in common with his fellow instructors, the high-school teacher of mathematics should have a special training in the particular line in which he is to give instruction. This special training should be sufficiently extensive to make him not only a master of his own subject, but also familiar with the other subjects with which his is directly related. The question is therefore quite germane to our discussion as to what should be the minimum preparation demanded of a high-school teacher of mathematics in his own and kindred lines that we may be insured of the best results.

It needs no argument to show that any teacher should know more of his subject than he is called upon to teach. School authorities are everywhere more and more recognizing the advantage of having teachers in charge of the various departments who are to some degree specialists in their respective fields. It is well that it is so, providing that it does not mean at the same time a narrowing of the intellectual horizon and training of the instructor. For some reason, however, we seem to have been slower to recognize the desirability, if not the necessity, of specially trained high-school teachers of mathematics than of the natural sciences and the languages. The opinion seems to be still entertained in many quarters that almost anyone will do to teach mathematics. This is clearly an error. On the other hand, we should not demand the same amount of special training

of a high-school teacher as would be expected of a college instructor in the same line. What, then, should be our standard? First of all, our teacher should know the possibilities and the limitations of public-school work, including the work of the grades, to the end that he may put himself in touch and sympathy with his pupils as they enter the high school. The instructor in charge of the mathematics in the high school should be the superintendent's most competent adviser concerning the mathematical work in the grades. If he is not familiar with this work, he should embrace the first opportunity to become so. His training in the higher mathematics should be sufficiently extended to include those branches for which his work directly prepares. In order to be able to point out to his pupils the significance of mathematics in its relation to the study of other branches of science, it is desirable that his training should include work in the applied as well as the pure mathematics. Such a preparation should include in pure mathematics courses in college algebra, analytical geometry, differential and integral calculus, and the theory of equations. In applied mathematics, he should have taken college work in general physics, mechanics, and perhaps astronomy. The work here outlined would require two years of college work in pure mathematics and perhaps a year and a half in the allied subjects. This is suggested as a *minimum* preparation for the instructor placed at the head of the mathematical department of a large high school. More is desirable. He would find it of advantage to have studied descriptive and modern geometry, to know something of the theory of functions with its applications to the mathematical physics, and to be familiar with the history of elementary mathematics. Many will regard this standard no doubt as too high, and will easily recall instances where teachers, who have not had the equivalent of this training, have nevertheless been regarded as efficient instructors of mathematics. However true this may be, it must be acknowledged that the character of instruction given by those same teachers would be improved rather than otherwise by the mathematical outlook which this training would have given them. There would be none of the hesitating and labored pro-

cedure so noticeable with the teacher not a complete master of the situation. There would be a definiteness of purpose in the work given and a familiarity with mathematical facts and principles which would inspire confidence, arouse enthusiasm, and show at once how real and important an element in a practical as well as a liberal education the study of mathematics is. There would be no false conceptions given, to be unlearned when the pupil enters college. Under such direction, the interest and spirit with which the student would take up his work in freshman mathematics would be greatly increased, for after all this is in a large measure the result of his high-school training.

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